

Relative non-relativistic mechanics

G. SARDANASHVILY

Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia

Abstract. Dynamic equations of non-relativistic mechanics are written in covariant-coordinate form in terms of relative velocities and accelerations with respect to an arbitrary reference frame. The notions of the non-relativistic reference frame, inertial force, free motion equation, and inertial frame are discussed.

1 Introduction

We consider second order dynamic equations in time-dependent non-relativistic mechanics. A configuration space of time-dependent non-relativistic mechanics is a smooth fibre bundle

$$\pi : Q \rightarrow \mathbb{R}. \quad (1)$$

A second order dynamic equation on this configuration space is a closed subbundle of the second order jet bundle $J^2Q \rightarrow J^1Q$. Given bundle coordinates (t, q^i) on Q and the adapted coordinates $(t, q^i, q_t^i, q_{tt}^i)$ on J^2Q , such an equation takes the coordinate form

$$q_{tt}^i = \xi^i(t, q^j, q_t^j). \quad (2)$$

We aim to bring this equation into the form (41) maintained under bundle coordinate transformations and expressed in relative velocities and accelerations with respect to an arbitrary reference frame [2, 4]. Recall that a reference frame in non-relativistic mechanics is defined as a connection on the configuration bundle (1) [1, 3, 6, 7]. The notions of an inertial force, free motion equation, and inertial frame are discussed.

For instance, a dynamic equation is said to be a free motion equation if there exists a reference frame such that this equation reads

$$\bar{q}_{tt}^i = 0. \quad (3)$$

One can formulate the necessary criterion whether the dynamic equation (2) is a free motion equation, but not the sufficient one. With respect to an arbitrary reference frame, the free motion equation (3) takes the form (32). One can think of its right-hand side as being a general expression of an inertial force in non-relativistic mechanics.

Note that Hamiltonian time-dependent mechanics with respect to an arbitrary reference frame has been formulated [5, 7].

2 Fibre bundles over \mathbb{R}

Throughout the paper, a typical fibre M of the fibre bundle Q (1) is an m -dimensional manifold. The base \mathbb{R} of Q is parameterized by the Cartesian coordinates t possessing the transition functions $t' = t + \text{const.}$ It is provided with the standard vector field ∂_t and the standard one-form dt which are invariant under the coordinate transformations $t' = t + \text{const.}$ The same symbol dt also stands for any pull-back of the standard one-form dt onto fibre bundles over \mathbb{R} . Given bundle coordinates (t, q^i) on Q , we sometimes use the compact notation $(q^\lambda) = (q^i, q^0 = t)$ of them. Recall the notation $d_t = \partial_t + q_t^i \partial_i + q_{tt}^i \partial_i^2$ of the total derivative,

Let us point out some peculiarities of fibre bundles and jet manifolds over \mathbb{R} .

Since \mathbb{R} is contractible, any fibre bundle over \mathbb{R} is obviously trivial. Its different trivializations

$$\psi : Q \cong \mathbb{R} \times M \quad (4)$$

differ from each other in the projections $Q \rightarrow M$, while the fibration $Q \rightarrow \mathbb{R}$ is once for all. Every trivialization (4) yields the corresponding trivialization of the jet manifold $J^1 Q \cong \mathbb{R} \times TM$. There is the canonical imbedding

$$\lambda_1 : J^1 Q \hookrightarrow TQ, \quad \lambda_1 : (t, q^i, \dot{q}^i) \mapsto (t, q^i, \dot{t} = 1, \dot{q}^i = q_t^i), \quad (5)$$

of the affine jet bundle

$$\pi_0^1 : J^1 Q \rightarrow Q \quad (6)$$

to the tangent bundle TQ of Q . Hereafter, we identify the jet manifold $J^1 Q$ with its affine image in TQ , modelled over the vertical tangent bundle VQ of the fibre bundle of $Q \rightarrow \mathbb{R}$.

A connection Γ on a fibre bundle $Q \rightarrow \mathbb{R}$ is defined as a global section

$$\Gamma = dt \otimes (\Gamma = \partial_t + \Gamma^i \partial_i)$$

of the affine jet bundle (6). In view of the morphism λ_1 (5), it can be identified to a nowhere vanishing horizontal vector field

$$\Gamma = \partial_t + \Gamma^i \partial_i \quad (7)$$

on Q which is the horizontal lift of the standard vector field ∂_t on \mathbb{R} by means of this connection. Conversely, any vector field Γ on Q such that $dt|_\Gamma = 1$ defines a connection on $Q \rightarrow \mathbb{R}$. The range of a connection Γ (7) is the kernel of the first order differential operator

$$D_\Gamma : J^1 Q \xrightarrow{Q} VQ, \quad \dot{q}^i \circ D_\Gamma = q_t^i - \Gamma^i, \quad (8)$$

on Q called the covariant differential of Γ .

Proposition 1. Since a connection Γ on $Q \rightarrow \mathbb{R}$ is always flat, it defines an atlas of local constant trivializations of $Q \rightarrow \mathbb{R}$ such that the associated bundle coordinates (t, \bar{q}^i) on Q possess the time-independent transition functions, and $\Gamma = \partial_t$ with respect to these coordinates. Conversely, every atlas of local constant trivializations of the fibre bundle $Q \rightarrow \mathbb{R}$ determines a connection on $Q \rightarrow \mathbb{R}$ which is equal to ∂_t relative to this atlas [2, 4].

A connection Γ on a fibre bundle $Q \rightarrow \mathbb{R}$ is said to be complete if the horizontal vector field (7) is complete.

Proposition 2. Every trivialization of a fibre bundle $Q \rightarrow \mathbb{R}$ yields a complete connection on this fibre bundle. Conversely, every complete connection Γ on $Q \rightarrow \mathbb{R}$ defines its trivialization (4) such that the vector field (7) equals ∂_t relative to the bundle coordinates associated to this trivialization [2].

Let $J^1 J^1 Q$ be the repeated jet manifold of a fibre bundle $Q \rightarrow \mathbb{R}$ (1), provided with the adapted coordinates $(t, q^i, q_t^i, q_{(t)}^i, q_{tt}^i)$. It possesses two affine fibrations

$$\begin{aligned} \pi_{11} : J^1 J^1 Q &\rightarrow J^1 Q, & q_t^i \circ \pi_{11} &= q_t^i, \\ J_0^1 \pi_0^1 : J^1 J^1 Q &\rightarrow J^1 Q, & q_t^i \circ J_0^1 \pi_0^1 &= q_{(t)}^i, \end{aligned}$$

which are canonically isomorphic:

$$\pi_{11} \circ k = J_0^1 \pi_{01}, \quad q_t^i \circ k = q_{(t)}^i, \quad q_{(t)}^i \circ k = q_t^i, \quad q_{tt}^i \circ k = q_{tt}^i. \quad (9)$$

The sesquiholonomic jet manifold $\hat{J}^2 Q \subset J^1 J^1 Q$ and the second order jet manifold $J^2 Q \subset J^1 J^1 Q$ are isomorphic and coordinated by $(t, q^i, q_t^i, q_{tt}^i)$. The affine bundle $J^2 Q \rightarrow J^1 Q$ is modelled over the vertical tangent bundle

$$V_Q J^1 Q = J^1 Q \times_Q VQ \rightarrow J^1 Q \quad (10)$$

of the affine jet bundle $J^1 Q \rightarrow Q$. There is the imbedding

$$\begin{aligned} J^2 Q &\xrightarrow{\lambda_2} T J^1 Q \xrightarrow{T\lambda_1} V_Q TQ \cong T^2 Q \subset TTQ, \\ \lambda_2 : (t, q^i, q_t^i, q_{tt}^i) &\mapsto (t, q^i, q_t^i, \dot{t} = 1, \dot{q}^i = q_t^i, \dot{q}_t^i = q_{tt}^i), \end{aligned} \quad (11)$$

where $(t, q^i, \dot{t}, \dot{q}^i, \ddot{t}, \ddot{q}^i)$ are the holonomic coordinates on the double tangent bundle TTQ , by $V_Q TQ$ is meant the vertical tangent bundle of $TQ \rightarrow Q$, and $T^2 Q$ is a second order tangent space, given by the coordinate relation $\dot{t} = 1$.

By a second order connection ξ on a fibre bundle $Q \rightarrow \mathbb{R}$ (1) is meant a connection on the jet bundle $J^1 Q \rightarrow \mathbb{R}$. Due to the imbedding (11), it is represented by a horizontal vector field

$$\xi = \partial_t + \chi_t^i \partial_i + \xi^i \partial_i^t$$

on $J^1 Q$ such that $\xi \rfloor dt = 1$. A second order connection which lives in $J^2 Q \subset J^1 J^1 Q$ is called holonomic. It reads

$$\xi = \partial_t + q_t^i \partial_i + \xi^i \partial_i^t. \quad (12)$$

Its range is the kernel of the covariant differential

$$D_\xi : J^1 J^1 Q \xrightarrow{J^1 Q} V_Q J^1 Q, \quad \dot{q}^i \circ D_\xi = 0, \quad \dot{q}_t^i \circ D_\xi = q_{tt}^i - \xi^i. \quad (13)$$

Every connection Γ on a fibre bundle $Q \rightarrow \mathbb{R}$ admits the jet prolongation to a section $J^1 \Gamma$ of the affine bundle $J^1 \pi_0^1$ and, by virtue of the isomorphism k (9), gives rise to the second order connection

$$J\Gamma = k \circ J^1 \Gamma : J^1 Q \rightarrow J^1 J^1 Q, \quad J\Gamma = \partial_t + \Gamma^i \partial_i + d_t \Gamma^i \partial_i^t. \quad (14)$$

3 Dynamic equations

A second order dynamic equation (or, simply, a dynamic equation) on a fibre bundle $Q \rightarrow \mathbb{R}$, by definition, is the range of a section of the jet bundle $J^2Q \rightarrow J^1Q$, i.e., a holonomic second order connection ξ (12) on $Q \rightarrow \mathbb{R}$ [2, 4]. This equation is the kernel of the covariant differential D_ξ (13) given by the coordinate equalities (2). The corresponding horizontal vector field ξ (12) is also called a dynamic equation. One can easily find the transformation law

$$q_{tt}^i = \xi^i, \quad \xi^i = (\xi^j \partial_j + q_t^j q_t^k \partial_j \partial_k + 2q_t^j \partial_j \partial_t + \partial_t^2) q^i(t, q^j) \quad (15)$$

of a dynamic equation under coordinate transformations $q^i \rightarrow q^i(t, q^j)$. By a solution of the dynamic equation (2) is meant a section of $Q \rightarrow \mathbb{R}$ whose second order jet prolongation lives in (2).

The fact that ξ (12) is a curvature-free connection places a limit on the geometric analysis of dynamic equations by holonomic second order connections. Therefore, we consider the relationship between the holonomic connections on the jet bundle $J^1Q \rightarrow \mathbb{R}$ and the connections on the affine jet bundle $J^1Q \rightarrow Q$ [2, 4]. The first order jet manifold of $J^1Q \rightarrow Q$ is denoted by $J_Q^1 J^1Q$.

Let $\gamma : J^1Q \rightarrow J_Q^1 J^1Q$ be a connection on $J^1Q \rightarrow Q$. It takes the coordinate form

$$\gamma = dq^\lambda \otimes (\partial_\lambda + \gamma_\lambda^i \partial_i^t). \quad (16)$$

Let us consider the composite fibre bundle

$$J^1Q \rightarrow Q \rightarrow \mathbb{R}. \quad (17)$$

There is the canonical morphism

$$\varrho : J_Q^1 J^1Q \ni (q^\lambda, q_t^i, q_{\lambda t}^i) \mapsto (q^\lambda, q_t^i, q_{(t)}^i = q_t^i, q_{tt}^i = q_{0t}^i + q_t^j q_{jt}^i) \in J^2Q.$$

Proposition 3. Any connection γ (16) on the affine jet bundle $J^1Q \rightarrow Q$ defines the second order holonomic connection

$$\xi_\gamma = \varrho \circ \gamma = \partial_t + q_t^i \partial_i + (\gamma_0^i + q_t^j \gamma_j^i) \partial_i^t. \quad (18)$$

It follows that every connection γ (16) on the affine jet bundle $J^1Q \rightarrow Q$ yields the dynamic equation

$$q_{tt}^i = \gamma_0^i + q_t^j \gamma_j^i. \quad (19)$$

on $Q \rightarrow \mathbb{R}$ which is the kernel, restricted to J^2Q , of the vertical covariant differential

$$\widetilde{D}_\gamma : J^1 J^1Q \rightarrow V_Q J^1Q, \quad \dot{q}^i \circ \widetilde{D}_\gamma = q_{tt}^i - \gamma_0^i - q_t^j \gamma_j^i, \quad (20)$$

of a connection γ . Therefore, connections on the jet bundle $J^1Q \rightarrow Q$ are called dynamic connections. A converse of Proposition 3 is the following.

Proposition 4. Any holonomic connection ξ (12) on the jet bundle $J^1Q \rightarrow \mathbb{R}$ yields the dynamic connection

$$\gamma_\xi = dt \otimes [\partial_t + (\xi^i - \frac{1}{2}q_t^j \partial_j^t \xi^i) \partial_i^t] + dq^j \otimes [\partial_j + \frac{1}{2} \partial_j^t \xi^i \partial_i^t]. \quad (21)$$

It is readily observed that the dynamic connection γ_ξ (21) possesses the property

$$\gamma_i^k = \partial_i^t \gamma_0^k + q_t^j \partial_i^t \gamma_j^k \quad (22)$$

which implies the relation $\partial_j^t \gamma_i^k = \partial_i^t \gamma_j^k$. Therefore, a dynamic connection γ , obeying the condition (22), is said to be *symmetric*. The torsion of a dynamic connection γ is defined as the tensor field

$$T = T_i^k \bar{d}q^i \otimes \partial_k : J^1Q \rightarrow V^*Q \otimes_Q VQ, \quad T_i^k = \gamma_i^k - \partial_i^t \gamma_0^k - q_t^j \partial_i^t \gamma_j^k. \quad (23)$$

It follows at once that a dynamic connection is symmetric iff its torsion vanishes. Let γ be a dynamic connection (16) and ξ_γ the corresponding dynamic equation (18). Then the dynamic connection (21) associated to the dynamic equation ξ_γ takes the form

$$\gamma_{\xi_\gamma i}^k = \frac{1}{2}(\gamma_i^k + \partial_i^t \gamma_0^k + q_t^j \partial_i^t \gamma_j^k), \quad \gamma_{\xi_\gamma 0}^k = \xi^k - q_t^i \gamma_{\xi_\gamma i}^k.$$

It is readily observed that $\gamma = \gamma_{\xi_\gamma}$ iff the torsion T (23) of the dynamic connection γ vanishes.

For instance, the affine jet bundle $J^1Q \rightarrow Q$ admits an affine connection

$$\gamma = dq^\lambda \otimes [\partial_\lambda + (\gamma_{\lambda 0}^i(q^\mu) + \gamma_{\lambda j}^i(q^\mu) q_t^j) \partial_i^t]. \quad (24)$$

This connection is symmetric iff $\gamma_{\lambda\mu}^i = \gamma_{\mu\lambda}^i$. One can easily justify that an affine dynamic connection generates a quadratic dynamic equation, and *vice versa*. Nevertheless, a non-affine dynamic connection, whose symmetric part is affine, also yields a quadratic dynamic equation.

4 Reference frames

From the physical viewpoint, a reference frame in non-relativistic mechanics determines a tangent vector at each point of a configuration space Q , which characterizes the velocity of an "observer" at this point. This speculation leads to the following notion of a reference frame in non-relativistic mechanics [1, 3, 6, 7].

Definition 5. In non-relativistic mechanics, a reference frame is a connection Γ on the configuration bundle $Q \rightarrow \mathbb{R}$.

In accordance with this definition, the corresponding covariant differential

$$\dot{q}_\Gamma^i = D_\Gamma(q_t^i) = q_t^i - \Gamma^i$$

determines the relative velocities with respect to the reference frame Γ .

By virtue of Proposition 1, any reference frame Γ on a configuration bundle $Q \rightarrow \mathbb{R}$ is associated to an atlas of local constant trivializations, and *vice versa*. The connection Γ reduces to $\Gamma = \partial_t$ with respect to the corresponding coordinates (t, \bar{q}^i) , whose transition functions $\bar{q}^i \rightarrow \bar{q}^i$ are independent of time. One can think of these coordinates as being also the reference frame, corresponding to the connection $\Gamma = \partial_t$. They are called the adapted coordinates to the reference frame Γ or, simply, a reference frame. In particular, with respect to the coordinates \bar{q}^i adapted to a reference frame Γ , the velocities relative to this reference frame are equal to the absolute ones

$$D_\Gamma(\bar{q}_t^i) = \dot{\bar{q}}_t^i = \bar{q}_t^i.$$

A reference frame is said to be complete if the associated connection Γ is complete. By virtue of Proposition 2, every complete reference frame defines a trivialization of a bundle $Q \rightarrow \mathbb{R}$, and *vice versa*.

Given a reference frame Γ , one should solve the equations

$$\Gamma^i(t, q^j(t, \bar{q}^a)) = \frac{\partial q^i(t, \bar{q}^a)}{\partial t}, \quad (25a)$$

$$\frac{\partial \bar{q}^a(t, q^j)}{\partial q^i} \Gamma^i(t, q^j) + \frac{\partial \bar{q}^a(t, q^j)}{\partial t} = 0 \quad (25b)$$

in order to find the coordinates (t, \bar{q}^a) adapted to Γ . Let (t, q_1^a) and (t, q_2^i) be the adapted coordinates for reference frames Γ_1 and Γ_2 , respectively. In accordance with the equality (25b), the components Γ_1^i of the connection Γ_1 with respect to the coordinates (t, q_2^i) and the components Γ_2^a of the connection Γ_2 with respect to the coordinates (t, q_1^a) fulfill the relation

$$\frac{\partial q_1^a}{\partial q_2^i} \Gamma_1^i + \Gamma_2^a = 0.$$

Using the relations (25a) – (25b), one can rewrite the coordinate transformation law (15) of dynamic equations as follows. Let

$$\bar{q}_{tt}^a = \bar{\xi}^a \quad (26)$$

be a dynamic equation on a configuration space Q , written with respect to a reference frame (t, \bar{q}^n) . Then, relative to arbitrary bundle coordinates (t, q^i) on $Q \rightarrow \mathbb{R}$, the dynamic equation (26) takes the form

$$q_{tt}^i = d_t \Gamma^i + \partial_j \Gamma^i (q_t^j - \Gamma^j) - \frac{\partial q^i}{\partial \bar{q}^a} \frac{\partial \bar{q}^a}{\partial q^j \partial q^k} (q_t^j - \Gamma^j) (q_t^k - \Gamma^k) + \frac{\partial q^i}{\partial \bar{q}^a} \bar{\xi}^a, \quad (27)$$

where Γ is the connection corresponding to the reference frame (t, \bar{q}^n) . The dynamic equation (27) can be expressed in the relative velocities $\dot{q}_t^i = q_t^i - \Gamma^i$ with respect to the initial

reference frame (t, \bar{q}^a) . We have

$$d_t \dot{q}_\Gamma^i = \partial_j \Gamma^i \dot{q}_\Gamma^j - \frac{\partial q^i}{\partial \bar{q}^a} \frac{\partial \bar{q}^a}{\partial q^j \partial q^k} \dot{q}_\Gamma^j \dot{q}_\Gamma^k + \frac{\partial q^i}{\partial \bar{q}^a} \bar{\xi}^a(t, q^j, \dot{q}_\Gamma^j). \quad (28)$$

Accordingly, any dynamic equation (2) can be expressed in the relative velocities $\dot{q}_\Gamma^i = \dot{q}_t^i - \Gamma^i$ with respect to an arbitrary reference frame Γ as follows:

$$d_t \dot{q}_\Gamma^i = (\xi - J\Gamma)_t^i = \xi^i - d_t \Gamma, \quad (29)$$

where $J\Gamma$ is the jet prolongation (14) of the connection Γ onto $J^1Q \rightarrow \mathbb{R}$.

Let us consider the following particular reference frame Γ for a dynamic equation ξ . The covariant differential of a reference frame Γ with respect to the corresponding dynamic connection γ_ξ (21) reads

$$\begin{aligned} \nabla^\gamma \Gamma &= \nabla_\lambda^\gamma \Gamma^k dq^\lambda \otimes \partial_k : Q \rightarrow T^*Q \times V_Q J^1Q, \\ \nabla_\lambda^\gamma \Gamma^k &= \partial_\lambda \Gamma^k - \gamma_\lambda^k \circ \Gamma. \end{aligned} \quad (30)$$

A connection Γ is called a geodesic reference frame for the dynamic equation ξ if

$$\Gamma \rfloor \nabla^\gamma \Gamma = \Gamma^\lambda (\partial_\lambda \Gamma^k - \gamma_\lambda^k \circ \Gamma) = (d_t \Gamma^i - \xi^i \circ \Gamma) \partial_i = 0. \quad (31)$$

It is readily observed that integral sections of a reference frame Γ are solutions of a dynamic equation ξ iff Γ is a geodesic reference frame for ξ .

5 Free motion equations

We have called the dynamic equation (2) the free motion equation if there exists a reference frame (t, \bar{q}^i) on the configuration bundle Q such that this equation takes the form (3). With respect to arbitrary bundle coordinates (t, q^i) , a free motion equation reads

$$q_{tt}^i = d_t \Gamma^i + \partial_j \Gamma^i (q_t^j - \Gamma^j) - \frac{\partial q^i}{\partial \bar{q}^m} \frac{\partial \bar{q}^m}{\partial q^j \partial q^k} (q_t^j - \Gamma^j) (q_t^k - \Gamma^k), \quad (32)$$

where $\Gamma^i = \partial_t q^i(t, \bar{q}^j)$ is the connection associated to the initial reference frame (t, \bar{q}^i) . One can think of the right-hand side of the equation (32) as being a general expression of an inertial force in non-relativistic mechanics. The corresponding dynamic connection γ_ξ on the affine jet bundle $J^1Q \rightarrow Q$ is

$$\gamma_k^i = \partial_k \Gamma^i - \frac{\partial q^i}{\partial \bar{q}^m} \frac{\partial \bar{q}^m}{\partial q^j \partial q^k} (q_t^j - \Gamma^j), \quad \gamma_0^i = \partial_t \Gamma^i + \partial_j \Gamma^i q_t^j - \gamma_k^i \Gamma^k. \quad (33)$$

Then, we come to the following criterion whether a dynamic equation is a free motion equation [2].

Proposition 6. If ξ is a free motion equation, then the curvature of the corresponding dynamic connection γ_ξ equals 0.

This criterion fails to be sufficient. If the curvature of a dynamic connection γ_ξ vanishes, it may happen that components of γ_ξ equal 0 with respect to non-holonomic bundle coordinates on the affine jet bundle $J^1Q \rightarrow Q$.

Note also that the dynamic connection (33) is affine. It follows that, if ξ is a free motion equation, it is always quadratic.

The free motion equation (32) is simplified if the coordinate transition functions $\bar{q}^i \rightarrow q^i$ are affine in coordinates \bar{q}^i . Then we have

$$q_{tt}^i = \partial_t \Gamma^i - \Gamma^j \partial_j \Gamma^i + 2q_t^j \partial_j \Gamma^i. \quad (34)$$

The following shows that the free motion equation (34) is affine in the coordinates q^i and q_t^i [2].

Proposition 7. Let (t, \bar{q}^a) be a reference frame on a configuration bundle $Q \rightarrow \mathbb{R}$ and Γ the corresponding connection. Components Γ^i of this connection with respect to another coordinate system (t, q^i) are affine functions of coordinates q^i iff the transition functions between the coordinates \bar{q}^a and q^i are affine.

The geodesic reference frames for a free motion equation are called inertial. They are $\Gamma^i = v^i = \text{const.}$ By virtue of Proposition 7, these reference frames define the adapted coordinates

$$\bar{q}^i = k_j^i q^j - v^i t - a^i, \quad k_j^i = \text{const.}, \quad v^i = \text{const.}, \quad a^i = \text{const.} \quad (35)$$

The equation (3) keeps obviously its free motion form under the transformations (35) between the geodesic reference frames. It is readily observed that these transformations are precisely the elements of the Galilei group.

6 Relative acceleration

It should be emphasized that, taken separately, the left- and right-hand sides of the dynamic equation (29) are not well-behaved objects. This equation can be brought into the covariant form if we introduce the notion of a relative acceleration.

To consider a relative acceleration with respect to a reference frame Γ , one should prolong the connection Γ on the configuration bundle $Q \rightarrow \mathbb{R}$ to a holonomic connection ξ_Γ on the jet bundle $J^1Q \rightarrow \mathbb{R}$. Note that the jet prolongation $J\Gamma$ (14) of Γ onto $J^1Q \rightarrow \mathbb{R}$ is not holonomic. We can construct the desired prolongation by means of a dynamic connection γ on the affine jet bundle $J^1Q \rightarrow Q$ [2].

Proposition 8. Let us consider the composite bundle (17). Given a frame Γ on $Q \rightarrow \mathbb{R}$ and a dynamic connections γ on $J^1Q \rightarrow Q$, there exists a dynamic connection $\tilde{\gamma}$ on $J^1Q \rightarrow Q$ with the components

$$\tilde{\gamma}_k^i = \gamma_k^i, \quad \tilde{\gamma}_0^i = d_t \Gamma^i - \gamma_k^i \Gamma^k. \quad (36)$$

We now construct a certain soldering form on the affine jet bundle $J^1Q \rightarrow Q$, and add it to this connection. Let us apply the canonical projection $T^*Q \rightarrow V^*Q$ and then the imbedding $\Gamma : V^*Q \rightarrow T^*Q$ to the covariant differential (30) of the reference frame Γ with respect to the dynamic connection γ . We obtain the $V_Q J^1Q$ -valued 1-form

$$\sigma = [-\Gamma^i(\partial_i \Gamma^k - \gamma_i^k \circ \Gamma)dt + (\partial_i \Gamma^k - \gamma_i^k \circ \Gamma)dq^i] \otimes \partial_k^t$$

on Q whose pull-back onto J^1Q is the desired soldering form. The sum $\gamma_\Gamma = \tilde{\gamma} + \sigma$, called the frame connection, reads

$$\gamma_{\Gamma 0}^i = d_t \Gamma^i - \gamma_k^i \Gamma^k - \Gamma^k(\partial_k \Gamma^i - \gamma_k^i \circ \Gamma), \quad \gamma_{\Gamma k}^i = \gamma_k^i + \partial_k \Gamma^i - \gamma_k^i \circ \Gamma. \quad (37)$$

This connection yields the desired holonomic connection

$$\xi_\Gamma^i = d_t \Gamma^i + (\partial_k \Gamma^i + \gamma_k^i - \gamma_k^i \circ \Gamma)(q_t^k - \Gamma^k)$$

on the jet bundle $J^1Q \rightarrow \mathbb{R}$.

Let ξ be a dynamic equation and $\gamma = \gamma_\xi$ the connection (21) associated to ξ . Then one can think of the vertical vector field

$$a_\Gamma = \xi - \xi_\Gamma = (\xi^i - \xi_\Gamma^i)\partial_i^t \quad (38)$$

on the affine jet bundle $J^1Q \rightarrow Q$ as being a relative acceleration with respect to the reference frame Γ in comparison with the absolute acceleration ξ .

For instance, let us consider a reference frame Γ which is geodesic for the dynamic equation ξ , i.e., the relation (31) holds. Then the relative acceleration with respect to the reference frame Γ is

$$(\xi - \xi_\Gamma) \circ \Gamma = 0.$$

Let ξ now be an arbitrary dynamic equation, written with respect to coordinates (t, q^i) adapted to the reference frame Γ , i.e., $\Gamma^i = 0$. In these coordinates, the relative acceleration with respect to a reference frame Γ is

$$a_\Gamma^i = \xi^i(t, q^j, q_t^j) - \frac{1}{2}q_t^k(\partial_k \xi^i - \partial_k \xi^i|_{q_t^j=0}). \quad (39)$$

Given another bundle coordinates (t, q'^i) on $Q \rightarrow \mathbb{R}$, this dynamic equation takes the form (28), while the relative acceleration (39) with respect to the reference frame Γ reads $a_\Gamma'^i = \partial_j q'^i a_\Gamma^j$. Then we can write a dynamic equation (2) in the form which is covariant under coordinate transformations, namely,

$$\widetilde{D}_{\gamma_\Gamma} q_t^i = d_t q_t^i - \xi_\Gamma^i = a_\Gamma, \quad (40)$$

where $\widetilde{D}_{\gamma_\Gamma}$ is the vertical covariant differential (20) with respect to the frame connection γ_Γ (37) on the affine jet bundle $J^1Q \rightarrow Q$.

In particular, if ξ is a free motion equation which takes the form (3) with respect to a reference frame Γ , then

$$\widetilde{D}_{\gamma_\Gamma} q_t^i = 0$$

relative to arbitrary bundle coordinates on the configuration bundle $Q \rightarrow \mathbb{R}$.

The left-hand side of the dynamic equation (40) can also be expressed in the relative velocities such that this dynamic equation takes the form

$$d_t \dot{q}_\Gamma^i - \gamma_{\Gamma k}^i \dot{q}_\Gamma^k = a_\Gamma, \quad (41)$$

which is the covariant form of the equation (29).

The concept of a relative acceleration is understood better when we deal with a quadratic dynamic equation ξ , and the corresponding dynamic connection γ is affine. If a dynamic connection γ is affine, i.e.,

$$\gamma_\lambda^i = \gamma_{\lambda 0}^i + \gamma_{\lambda k}^i q_t^k,$$

so is a frame connection γ_Γ for any frame Γ :

$$\begin{aligned} \gamma_{\Gamma jk}^i &= \gamma_{jk}^i, \\ \gamma_{\Gamma 0k}^i &= \partial_k \Gamma^i - \gamma_{jk}^i \Gamma^j, \quad \gamma_{\Gamma k0}^i = \partial_k \Gamma^i - \gamma_{kj}^i \Gamma^j, \\ \gamma_{\Gamma 00}^i &= \partial_t \Gamma^i - \Gamma^j \partial_j \Gamma^i + \gamma_{jk}^i \Gamma^j \Gamma^k. \end{aligned} \quad (42)$$

In particular, we obtain

$$\gamma_{\Gamma jk}^i = \gamma_{jk}^i, \quad \gamma_{\Gamma 0k}^i = \gamma_{\Gamma k0}^i = \gamma_{\Gamma 00}^i = 0$$

relative to the coordinates adapted to a reference frame Γ . A glance at the expression (42) shows that, if a dynamic connection γ is symmetric, so is a frame connection γ_Γ . Thus, we come to the following.

Proposition 9. If a dynamic equation ξ is quadratic, the relative acceleration a_Γ (38) is always affine, and it admits the decomposition

$$a_\Gamma^i = -(\Gamma^\lambda \nabla_\lambda^\gamma \Gamma^i + 2\dot{q}_\Gamma^\lambda \nabla_\lambda^\gamma \Gamma^i), \quad (43)$$

where $\gamma = \gamma_\xi$ is the dynamic connection (21), and

$$\dot{q}_\Gamma^\lambda = q_t^\lambda - \Gamma^\lambda, \quad q_t^0 = 1, \quad \Gamma^0 = 1,$$

is the relative velocity with respect to the reference frame Γ .

Note that the splitting (43) provides a generalized Coriolis theorem. In particular, the well-known analogy between inertial and electromagnetic forces is restated. Proposition 9 shows that this analogy can be extended to an arbitrary quadratic dynamic equation.

References

- [1] A.Echeverría Enríquez, M.Muñoz Lecanda and N.Román Roy, Non-standard connections in classical mechanics, *J. Phys. A* **28** (1995) 5553.
- [2] L.Mangiarotti and G.Sardanashvily, *Gauge Mechanics* (World Scientific, Singapore, 1998).
- [3] L.Mangiarotti and G.Sardanashvily, On the geodesic form of second order dynamic equations, *J. Math. Phys.* **41** (2000) 835.
- [4] L.Mangiarotti and G.Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific, Singapore, 2000).
- [5] L.Mangiarotti and G.Sardanashvily, Quantum mechanics with respect to different reference frames, *J. Math. Phys.* **48** (2007) 082104.
- [6] E.Massa and E.Pagani, Jet bundle geometry, dynamical connections and the inverse problem of Lagrangian mechanics, *Ann. Inst. Henri Poincaré* **61** (1994) 17.
- [7] G.Sardanashvily, Hamiltonian time-dependent mechanics, *J. Math. Phys.* **39** (1998) 2714.